The Einstein 3-form G_α and its equivalent 1-form L_α in Riemann-Cartan space

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Abstract

The definition of the Einstein 3-form G_α is motivated by means of the contracted 2nd Bianchi identity. This definition involves at first the complete curvature 2-form. The 1-form L_α is defined via $G_\alpha = L^\beta \wedge {}^\star(\vartheta_\beta \wedge \vartheta_\alpha)$ (here * is the Hodge-star, ϑ_α the coframe). It is equivalent to the Einstein 3-form and represents a certain contraction of the curvature 2-form. A variational formula of Salgado on quadratic invariants of the L_α 1-form is discussed, generalized, and put into proper perspective.

1 Introduction

In a Riemannian space, the curvature can be split into the conformal (Weyl) curvature and a piece which contains a 1-form, here called L_α. That 1-form has some very interesting properties: (i) It is closely related to the Einstein 3-form. (ii) It plays an important role in the formulation of the initial value problem which focuses on symmetric-hyperbolic equations for the conformal curvature. (iii) Recently, we learned of a nice formula of Salgado [8] which involves an quadratic invariant of L_α. (iv) It plays a role in the context of current investigations of the Cotton 2-form C_α. The 1-form L_α appears as the potential of the Cotton 2-form, $C_\alpha := DL_\alpha$. The C_α is related to the conformal properties of space and, in 3 dimensions, substitutes the Weyl curvature in the criterion for conformal flatness.

These points motivate a closer look at L_α. We will investigate mainly its algebraic structure and generalize it to an n-dimensional Riemann-Cartan space. The differential properties, that is, the Cotton 2-form, will be in the center of interest of a forthcoming article [4].

In this article, we would like to cast some light on the Einstein $(n-1)$-form and related quantities. Within the framework of the calculus of exterior differential forms these structures will arise quite naturally.

In section 2 we introduce some notation and motivate the definition of the Einstein $(n-1)$-form. We give the well known "differential argument", involving the contracted 2nd Bianchi identity, and a less well-known algebraic argument.

In section 3 we derive two quantities equivalent to the Einstein 3-form, the Einstein tensor and the so-called L^α 1-form.

In section 4 we discuss an invariant containing L_α and G_α which was found by Salgado [8] and generalize it to a Riemann-Cartan space.

Section 5 puts the previous results into the context of the irreducible decomposition of the curvature.

We close with a remark concerning the role of the derivative of L_α, DL_α, which also is known as Cotton 2-form C_α.

2 Bianchi identities and the Einstein $(n-1)$-form

On a differentiable manifold of arbitrary dimension we start with a *coframe*[1]

$$\vartheta^\alpha = e_i{}^\alpha \, dx^i \,. \tag{1}$$

The coframe is called natural or holonomic if $e^\alpha{}_i = \delta^\alpha_i$. The vector basis or frame which is dual to this particular coframe is denoted by e_α,

$$e_\alpha = e^i{}_\alpha \partial_i \,, \quad e_\alpha \rfloor \vartheta^\beta = \delta^\alpha_\beta \,. \tag{2}$$

We then may introduce a *connection* 1-form

$$\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta \, dx^i \,. \tag{3}$$

Thereby we define the exterior covariant derivative of a tensor-valued p-form (d denotes the exterior derivative)

$$D\phi_{\alpha\,\ldots}{}^{\beta\,\cdots} := d\phi_{\alpha\,\ldots}{}^{\beta\,\cdots} - \Gamma_\alpha{}^\gamma \wedge \phi_{\gamma\,\ldots}{}^{\beta\,\cdots} + \Gamma_\gamma{}^\beta \wedge \phi_{\alpha\,\ldots}{}^{\gamma\,\cdots} \,. \tag{4}$$

Subsequently, we define the torsion, a vector-valued two-form T^α by

$$T^\alpha = \frac{1}{2} T_{ij}{}^\alpha \, dx^i \wedge dx^j := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta \,, \ 1^{\text{st}} \text{ structure eq.}\,, \tag{5}$$

and the curvature, an antisymmetric 2-form $R_\alpha{}^\beta$ by

$$R_\alpha{}^\beta = \frac{1}{2} R_{ij\alpha}{}^\beta \, dx^i \wedge dx^j := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta \,, \quad 2^{\text{nd}} \text{ structure eq.} \tag{6}$$

From these definitions, together with that of the covariant exterior derivative, we can deduce the following two identities:

$$DT^\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta \,, \qquad 1^{\text{st}} \text{ Bianchi identity}\,, \tag{7}$$

$$DR_\alpha{}^\beta = 0 \,, \qquad 2^{\text{nd}} \text{ Bianchi identity}\,. \tag{8}$$

[1] We use Latin letters for holonomic and Greek letters for anholonomic indices.

Supplied with a *metric* g and the corresponding Hodge-star duality operator *, we can define the η-*basis*[2]

$$
\begin{array}{llllll}
\eta & := & {}^\star 1\,, & & & \text{basis of n-forms,}\\
\eta_{\alpha_1} & := & {}^\star\vartheta_{\alpha_1} & = & e_{\alpha_1}\rfloor\eta\,, & \text{basis of } (n-1)\text{-forms,}\\
\eta_{\alpha_1\alpha_2} & := & {}^\star(\vartheta_{\alpha_1}\wedge\vartheta_{\alpha_2}) & = & e_{\alpha_2}\rfloor\eta_{\alpha_1}\,, & \text{basis of } (n-2)\text{-forms,}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\eta_{\alpha_1\alpha_2\ldots\alpha_n} & := & {}^\star(\vartheta_{\alpha_1}\wedge\vartheta_{\alpha_2}\wedge\ldots\wedge\vartheta_{\alpha_n}) & = & e_{\alpha_n}\rfloor\eta_{\alpha_1\alpha_2\ldots\alpha_{(n-1)}}\,. &
\end{array}
\tag{9}
$$

If we require metric-compatibility of the connection, i. e., $Dg_{\alpha\beta}=0$, we arrive at a Riemann-Cartan space. In orthonormal frames, we find $\Gamma^{\alpha\beta}=-\Gamma^{\beta\alpha}$.

By contracting the second Bianchi identity (8) twice, we find

$$
e_\beta\rfloor e_\alpha\rfloor DR^{\alpha\beta}=0 \qquad \overset{\substack{\text{taking}\\ \text{Hodge-dual}}}{\Longrightarrow} \qquad \vartheta_\alpha\wedge\vartheta_\beta\wedge{}^\star DR^{\alpha\beta}=0\,. \tag{10}
$$

This corresponds to an irreducible piece of the second Bianchi identity, see [5]. For $n>3$, we obtain another differential identity of the curvature 2-form by taking the exterior products of eq.(10) and ϑ_γ. By using rule H5 for the Hodge-star (see appendix) we find

$$
DR^{\alpha\beta}\wedge\eta_{\alpha\beta\gamma}=0\,. \tag{11}
$$

By differentiation,

$$
D\left(R^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma}\right)=\left(DR^{\beta\gamma}\right)\wedge\eta_{\alpha\beta\gamma}+R^{\beta\gamma}\wedge D\eta_{\alpha\beta\gamma}\,, \tag{12}
$$

or, by (11) and $D\eta_{\alpha\beta\gamma}=T^\delta\wedge\eta_{\alpha\beta\gamma\delta}$ (see [6], eq.(3.8.5)),

$$
D\left(\frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge R^{\beta\gamma}\right)=\frac{1}{2}\,\eta_{\alpha\beta\gamma\delta}\,\wedge T^\beta\wedge R^{\gamma\delta}\,. \tag{13}
$$

In a Riemann space, where the torsion is zero, *and* in a Weitzenböck space, where the curvature is zero, the term on the right hand side vanishes.

Another interesting property of the $(n-1)$-form $\eta_{\alpha\beta\gamma}\wedge R^{\beta\gamma}$ should be mentioned here. From special relativity we know that the energy-momentum current density has to be represented by a vector-valued $(n-1)$-form. Suppose one has the idea to link energy-momentum to curvature. We then notice that the expression $\eta_{\alpha\beta\gamma}\wedge R^{\beta\gamma}$ is one of the two most obvious vector valued $(n-1)$-forms linear in the curvature. The other one is $\vartheta_\beta\wedge{}^\star R^{\alpha\beta}$. However, in a Riemannian space, only the first quantity is automatically conserved[3] which perfectly matches the conserved energy-momentum.[4]

[2] The η-*basis* seemingly was introduced by Trautman, see [11].

[3] In a *Riemannian* space we find for the covariant derivative of $\vartheta_\beta\wedge{}^\star R^{\alpha\beta}$, using the notation introduced in section 3, $D\left(\vartheta_\beta\wedge{}^\star R^{\alpha\beta}\right)=-D^\star\left(e_\beta\rfloor R^{\alpha\beta}\right)=-D^\star(\mathrm{Ric}_\nu{}^\alpha\,\vartheta^\nu)=-(\nabla_\mu\mathrm{Ric}_\nu{}^\alpha)\,\vartheta^\mu\wedge\eta^\nu=-(\nabla_\mu\mathrm{Ric}_\nu{}^\alpha)\,g^{\nu\mu}\,\eta=-\frac{1}{2}\,(\nabla^\alpha\mathrm{R})\,\eta$. The last expression, which is non-zero in general, is obtained by means of the 2^{nd} Bianchi identity.

[4] In a Riemann-Cartan space appear additional forces on the right-hand side of the energy-momentum law, such as the *Mathisson-Papapetrou force*. This is consistent with the non-vanishing right-hand side of eq.(13), see [5].

These considerations motivate the definition

$$\boxed{G_\alpha := \frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge R^{\beta\gamma}\,, \qquad \textit{Einstein } (n-1)\textit{-form.} \qquad (14)}$$

3 Alternative representations of the Einstein $(n-1)$-form

The $(n-1)$-form (14) naturally appears as a piece of the identically vanishing $(n+1)$-form $R^{\beta\gamma}\wedge\eta_\alpha$. We try to extract G_α from $R^{\beta\gamma}\wedge\eta_\alpha$ by contracting the latter twice:

$$\begin{aligned} 0 &= e_\gamma\rfloor e_\beta\rfloor\left(R^{\beta\gamma}\wedge\eta_\alpha\right) = e_\gamma\rfloor\left[\left(e_\beta\rfloor R^{\beta\gamma}\right)\wedge\eta_\alpha + R^{\beta\gamma}\wedge\eta_{\alpha\beta}\right] \\ &= \underbrace{\left(e_\gamma\rfloor e_\beta\rfloor R^{\beta\gamma}\right)}_{=-\mathrm{R}}\wedge\eta_\alpha - \underbrace{\left(e_\beta\rfloor R^{\beta\gamma}\right)}_{=-\mathrm{Ric}^\gamma}\wedge\eta_{\alpha\gamma} + \underbrace{\left(e_\gamma\rfloor R^{\beta\gamma}\right)}_{=\mathrm{Ric}^\beta}\wedge\eta_{\alpha\beta} + \underbrace{R^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma}}_{=2G_\alpha}\,, \end{aligned} \tag{15}$$

$$\Rightarrow\; G_\alpha = -\mathrm{Ric}^\beta\wedge\eta_{\alpha\beta} + \frac{1}{2}\,\mathrm{R}\,\eta_\alpha\,, \tag{16}$$

where we introduced the *Ricci 1-form* $\mathrm{Ric}_\alpha := e_\beta\rfloor R_\alpha{}^\beta$, and its trace, the *curvature scalar* $\mathrm{R} := e_\alpha\rfloor\mathrm{Ric}^\alpha$. In a Riemann-Cartan space we have $R^{\alpha\beta} = -R^{\beta\alpha}$. By means of the definition of the curvature 2-form, eq.(6), we find $\mathrm{Ric}_\alpha = \mathrm{Ric}_{\nu\alpha}\,\vartheta^\nu$, where $\mathrm{Ric}_{\nu\alpha} := R_{\mu\nu\alpha}{}^\mu$ denotes the Ricci tensor. It is symmetric if the torsion is covariantly constant as, for instance, in a Riemann space, where $T^\alpha \equiv 0$.

Often the Ricci tensor is defined by contraction of the 2nd and 4th index in our Schouten notation of the curvature tensor. Because of the antisymmetry of the curvature tensor, this definition of the Ricci tensor differs from our convention by a sign. This applies also to quantities which are derived from the Ricci tensor, like the Einstein tensor and the $L_{\alpha\beta}$ tensor.

Einstein tensor

Since G_α is a $(n-1)$-form, we can decompose it with respect to the basis of $(n-1)$-forms η_α:

$$G_\alpha =: G_\alpha{}^\beta\,\eta_\beta\,. \tag{17}$$

By convention, we contract here the 2nd index of $G_\alpha{}^\beta$. In order to determine the components $G_\alpha{}^\beta$, we just have to rewrite the first term of the right hand side of eq.(15):

$$\begin{aligned} \mathrm{Ric}^\beta\wedge\eta_{\alpha\beta} &= \mathrm{Ric}_\mu{}^\beta\,\vartheta^\mu\wedge\eta_{\alpha\beta} = \mathrm{Ric}_\mu{}^\beta\,\vartheta^\mu\wedge{}^\star(\vartheta_\alpha\wedge\vartheta_\beta) \\ &= -\mathrm{Ric}_\mu{}^\beta\,{}^\star\left(e^\mu\rfloor(\vartheta_\alpha\wedge\vartheta_\beta)\right) = -\mathrm{Ric}_\mu{}^\beta\,{}^\star(\delta^\mu_\alpha\,\vartheta_\beta - \vartheta_\alpha\,\delta^\mu_\beta) \\ &= -\mathrm{Ric}_\alpha{}^\beta\,\eta_\beta + \mathrm{R}\,\eta_\alpha\,. \end{aligned} \tag{18}$$

By substituting (18) into (15) we find

$$G_{\alpha\beta} = \mathrm{Ric}_{\alpha\beta} - \frac{1}{2}\,\mathrm{R}\,g_{\alpha\beta}\,. \tag{19}$$

Thereby we recover the usual definition of the Einstein tensor.

The L_α 1-form

Using the identity $\vartheta^\alpha \wedge (e_\alpha \rfloor \Phi) = {\scriptstyle (\mathrm{rank}\,\Phi)}\ \Phi$, we can rewrite η_α according to

$$(n-1)\,\eta_\alpha = \vartheta^\beta \wedge (e_\beta \rfloor \eta_\alpha) = \vartheta^\beta \wedge \eta_{\alpha\beta}\,. \tag{20}$$

Substituting this into (16), we arrive at

$$G_\alpha = \mathrm{Ric}^\beta \wedge \eta_{\beta\alpha} - \frac{1}{2(n-1)}\,\mathrm{R}\,\vartheta^\beta \wedge \eta_{\beta\alpha}\,. \tag{21}$$

This suggests the definition

$$\boxed{L^\beta := \mathrm{Ric}^\beta - \frac{1}{2(n-1)}\,\mathrm{R}\,\vartheta^\beta\,,} \tag{22}$$

such that

$$G_\alpha = L^\beta \wedge \eta_{\beta\alpha}\,. \tag{23}$$

The decomposition of L_α in components reads

$$L^\alpha =: L_\mu{}^\alpha\,\vartheta^\mu\,. \tag{24}$$

In a Riemann-Cartan space, the tensor

$$L_\mu{}^\alpha = \mathrm{Ric}_\mu{}^\alpha - \frac{1}{2(n-1)}\,\mathrm{R}\delta_\mu^\alpha\,, \tag{25}$$

if α is lowered, is *not* symmetric in general.

The trace of L_α is proportional to the curvature scalar

$$L := e_\alpha \rfloor L^\alpha = \frac{n-2}{2(n-1)}\,\mathrm{R}\,. \tag{26}$$

We collect the results of this section for a Riemann-Cartan space in

$$\boxed{\begin{array}{rcccl} G_\alpha & = & \frac{1}{2}\,R^{\beta\gamma} & \wedge & \eta_{\alpha\beta\gamma} \\ & = & L^\beta & \wedge & \eta_{\beta\alpha} \\ & = & G_\alpha{}^\beta & \wedge & \eta_\beta\,. \end{array}} \tag{27}$$

4 On Salgado's formula

Since G_α is a $(n-1)$-form and L^α a 1-form, $L^\alpha \wedge G_\alpha$ is a scalar-valued n-form and thus a possible candidate for a *Lagrange n-form*. Moreover, we can guess that the variation of $L^\alpha \wedge G_\alpha$ with respect to L^α should yield G_α, and vice versa. However, because L_α and G_α are not independent, we have to check this explicitly. By means of the results of the last section we find

$$\begin{aligned}
\delta \frac{1}{2} \left(L^\alpha \wedge G_\alpha \right) &= \frac{1}{2} \left(\delta L^\alpha \wedge G_\alpha + L^\alpha \wedge \delta G_\alpha \right) \\
&= \frac{1}{2} \left(\delta L^\alpha \wedge G_\alpha + L^\alpha \wedge \delta \left(L^\gamma \wedge \eta_{\gamma\alpha} \right) \right) \\
&= \frac{1}{2} \left(\delta L^\alpha \wedge G_\alpha + L^\alpha \wedge \delta L^\gamma \wedge \eta_{\gamma\alpha} + L^\alpha \wedge L^\gamma \wedge \delta \eta_{\gamma\alpha} \right) \\
&= \frac{1}{2} (\delta L^\alpha \wedge G_\alpha - \delta L^\gamma \wedge \underbrace{L^\alpha \wedge \eta_{\gamma\alpha}}_{=-G_\gamma} + L^\alpha \wedge L^\gamma \wedge \delta \eta_{\gamma\alpha}) \\
&= \delta L^\alpha \wedge G_\alpha - \frac{1}{2} L^\alpha \wedge L^\beta \wedge \delta \eta_{\alpha\beta} \,. \qquad (28)
\end{aligned}$$

Thus we have

$$\boxed{\frac{1}{2} \frac{\delta \left(L^\alpha \wedge G_\alpha \right)}{\delta L^\beta} = G_\beta \,. \qquad (29)}$$

This formula also becomes apparent by noticing that

$$\frac{1}{2} L^\alpha \wedge G_\alpha = -\frac{1}{2} L^\alpha \wedge L^\beta \wedge \eta_{\alpha\beta} \,. \qquad (30)$$

For the variation of $L^\alpha \wedge G_\alpha$ with respect to G_α we have to express L^α in terms of G_α. We start from

$$G_\alpha = L^\beta \wedge \eta_{\beta\alpha} = L_\mu{}^\beta \, \vartheta^\mu \wedge \eta_{\beta\alpha} \,, \qquad (31)$$

The term $\vartheta^\mu \wedge \eta_{\beta\alpha}$ can be rewritten as in eq.(18) yielding

$$G_\alpha = L_\alpha{}^\beta \, \eta_\beta - L \, \eta_\alpha = \left(L_\alpha{}^\beta - L \delta_\alpha^\beta \right) \eta_\beta \,. \qquad (32)$$

From this equation we infer for the traces L and $\overset{\star}{G}$

$$\overset{\star}{G} := e_\alpha \rfloor {}^\star G^\alpha = {}^\star (G^\alpha \wedge \vartheta_\alpha) = (-1)^{(n-1+\text{ind})} \, (1-n) \, L \,. \qquad (33)$$

The last two equations lead to

$$\begin{aligned}
L_\alpha{}^\mu &= L_\alpha{}^\beta \, \delta_\beta^\mu = L_\alpha{}^\beta \, e^\mu \rfloor \vartheta_\beta = L_\alpha{}^\beta \, e^\mu \rfloor \left((-1)^{(n-1+\text{ind})} \, {}^\star \eta_\beta \right) \\
&= (-1)^{(n-1+\text{ind})} \left(e^\mu \rfloor {}^\star G_\alpha - \frac{1}{n-1} \overset{\star}{G} \, \delta_\alpha^\mu \right) , \qquad (34)
\end{aligned}$$

or, by multiplying with ϑ^α and using the rules for the Hodge-dual,[5]

$$L^\mu = (-1)^{\text{ind}}\, {}^\star\Big[e^\alpha \rfloor (G_\alpha \wedge \vartheta^\mu) - \frac{1}{n-1}\, e^\mu \rfloor (G_\alpha \wedge \vartheta^\alpha)\Big]\,. \tag{35}$$

Since Hodge-star, interior and exterior products are linear, eq.(35) is linear in G_α. Consequently, the variation of L_α with respect to G_α reads

$$\delta L^\alpha(G_\beta) = L^\alpha(G_\beta + \delta G_\beta) - L^\alpha(G_\beta) = L^\alpha(\delta G_\beta)\,. \tag{36}$$

A simple, but somewhat lengthy, calculation shows[6]

$$L^\alpha(\delta G_\beta) \wedge G_\alpha = L^\alpha \wedge \delta G_\alpha\,. \tag{37}$$

Then the variation of $L^\alpha \wedge G_\alpha$ with respect to G_α turns out to be

$$\frac{1}{2}\,\delta\,(L^\alpha \wedge G_\alpha) = \frac{1}{2}\,\left[(\delta L^\alpha) \wedge G_\alpha + L^\alpha \wedge (\delta\, G_\alpha)\right] = L^\alpha \wedge \delta G_\alpha\,. \tag{38}$$

$$\boxed{\frac{1}{2}\,\frac{\delta\,(L^\alpha \wedge G_\alpha)}{\delta G_\beta} = (-1)^{(n-1)}\, L^\beta\,. \qquad (39)}$$

Component representation

Evaluating $e_\beta \rfloor e_\alpha \rfloor (L^\alpha \wedge L^\beta \wedge \eta) = 0$ yields

$$\begin{aligned} \frac{1}{2}\, L^\alpha \wedge G_\alpha &= -\frac{1}{2}\, L^\alpha \wedge L^\beta \wedge \eta_{\alpha\beta} = -\frac{1}{2}\left(L^\alpha{}_\alpha\, L^\beta{}_\beta - L^\alpha{}_\beta\, L^\beta{}_\alpha\right)\eta \\ &= -\left(L^\alpha{}_{[\alpha}\, L^\beta{}_{\beta]}\right)\eta\,, \end{aligned} \tag{40}$$

Eq.(29) corresponds to the Salgado formula [8]

$$\frac{d\left(L^\alpha{}_{[\alpha}\, L^\beta{}_{\beta]}\right)}{dL^\mu{}_\nu} = -L^\nu{}_\mu + \delta^\nu_\mu\, L = -G^\nu{}_\mu\,, \tag{41}$$

which was found by Salgado in a Riemannian context. It remains valid in a Riemann-Cartan space. Eqs.(19, 25) yield

$$L^\alpha{}_\beta = G^\alpha{}_\beta - \frac{1}{n-1}\, G^\mu{}_\mu\, \delta^\alpha_\beta\,. \tag{42}$$

Differentiating the last equation we get

$$\frac{\partial\, L^\alpha{}_\beta}{\partial\, G^\mu{}_\nu} = \delta^\alpha_\mu\, \delta^\nu_\beta - \frac{1}{n-1}\delta^\nu_\mu\, \delta^\alpha_\beta\,. \tag{43}$$

[5]If $DT^\alpha \equiv 0$, $L_{\alpha\beta}$ is symmetric and we simply have ${}^\star L^\alpha = {}^\star(L_\mu{}^\alpha\, \vartheta^\mu) = L_\mu{}^\alpha\, {}^\star\vartheta^\mu = L_\mu{}^\alpha\, \eta^\mu = L^\alpha{}_\mu\, \eta^\mu$, or, by substituting this into eq.(32), $G_\alpha = {}^\star L_\alpha - L\eta_\alpha$.

[6]See appendix.

Substituting eq.(42) into $L^{\alpha}{}_{[\alpha}\, L^{\beta}{}_{\beta]}$, we find

$$L^{\alpha}{}_{[\alpha}\, L^{\beta}{}_{\beta]} = -\frac{1}{2} L^{\alpha}{}_{\beta}\, G^{\beta}{}_{\alpha}\,. \tag{44}$$

From the last two equation we derive

$$\frac{d}{dG^{\mu}{}_{\nu}} \left(L^{\alpha}{}_{[\alpha}\, L^{\beta}{}_{\beta]} \right) = -L^{\nu}{}_{\mu}\,. \tag{45}$$

5 The 1-form L_α and the irreducible decomposition of the curvature

The 1-form L_α represents the trace-part (that is, the second rank pieces of a fourth rank quantity) $e_\beta \rfloor R^{\alpha\beta} = L^\alpha + \frac{1}{n-2} L \vartheta^\alpha$ of the curvature. This property seems to be nothing special because it also belongs to other contractions of the curvature (like the Ricci- and the Einstein-tensor). However, the Einstein tensor, which is a trace-modified Ricci-tensor, is an interesting quantity because of a property not shared by the Ricci-tensor, namely to be divergence-free. What are the properties peculiar to L_α?

In a Riemann-Cartan space, the 1-form L_α represents that part of the curvature 2-form which has the structure $\vartheta_{[\alpha} \wedge (1-\text{form})_{\beta]}$. To see this, one has to perform an irreducible decomposition of the curvature. We use the results obtained in [7] and find

$$\begin{array}{lcccccc}
\mathrm{R}_{\alpha\beta} & = & {}^{(1)}R_{\alpha\beta} & + & {}^{(2)}R_{\alpha\beta} + {}^{(3)}R_{\alpha\beta} & + & {}^{(4)}R_{\alpha\beta} + {}^{(5)}R_{\alpha\beta} + {}^{(6)}R_{\alpha\beta} \\
 & = & {}^{(1)}R_{\alpha\beta} & + & (-1)^{\mathrm{ind}}\,{}^\star\left(\vartheta_{[\alpha} \wedge P_{\beta]}\right) & - & \frac{2}{n-2}\vartheta_{[\alpha} \wedge L_{\beta]} \\
 & =: & \mathrm{Weyl}_{\alpha\beta} & + & \overset{P}{R}{}_{\alpha\beta} & + & \overset{L}{R}{}_{\alpha\beta} \\
 & = & \begin{array}{c}\text{irreducible}\\ 4^{\text{th}}\text{-rank}\end{array} & + & \text{pseudo-trace piece} & + & \text{trace piece}
\end{array} \tag{46}$$

The ${}^{(i)}R_{\alpha\beta}$ denote the 6 irreducible pieces of the curvature in a Riemann-Cartan space. For their precise definition we refer the reader to the literature, see [7, 6], e. g., because, in this context, the main result is contained in the second line of (46). The curvature decomposes into the conformal curvature $\mathrm{Weyl}_{\alpha\beta}$, a trace piece piece $\overset{L}{R}{}_{\alpha\beta}$, determined by L_α, and a pseudo-trace piece $\overset{P}{R}{}_{\alpha\beta}$, which is determined by a $(n-3)$-form[7] P_α. If $DT^\alpha \equiv 0$, i. e., in particular in a Riemannian space, we have, due to the 1st Bianchi identity (7), ${}^{(2)}R_{\alpha\beta} = {}^{(3)}R_{\alpha\beta} = {}^{(5)}R_{\alpha\beta} = 0$ and thus

$$R_{\alpha\beta} = \mathrm{Weyl}_{\alpha\beta} - \frac{2}{n-2}\, \vartheta_{[\alpha} \wedge L_{\beta]}\,, \qquad \text{if } T^\alpha \equiv 0. \tag{47}$$

[7]For the sake of completeness we display its explicit form: $P_\alpha := {}^\star(R^{\beta}{}_{\alpha} \wedge \vartheta_\beta) - \frac{1}{6}\, \vartheta_\alpha \wedge {}^\star(R^{\beta\gamma} \wedge \vartheta_\beta \wedge \vartheta_\gamma) - \frac{1}{n-2}\, e_\alpha \rfloor \left[\vartheta^\beta \wedge {}^\star(R^{\gamma}{}_{\beta} \wedge \vartheta_\gamma)\right]$

The corresponding formula in Ricci calculus is often used for defining the Weyl tensor $C_{\alpha\beta\gamma\delta}$. Eq.(47), decomposed into components reads

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{4}{n-2}\, g_{\left[\alpha\right|\left[\gamma\right.} L_{\left.\delta\right]\left|\beta\right]}\,, \qquad \text{if } T_{\alpha\beta}{}^{\gamma} \equiv 0. \tag{48}$$

In a Riemannian space, $\mathrm{Weyl}_{\alpha\beta}$ transforms like a conformal density. Thus, L_α represents that piece of the curvature which does not transform like a conformal density. This properties of L_α are well known, compare [2] and [9].

For the number of independent components we have

$$\mathrm{Weyl}_{\alpha\beta} \quad \rightarrow \quad \frac{1}{12}(n+2)(n+1)n(n-3)\,, \tag{49}$$

$$\overset{P}{R}_{\alpha\beta} \quad \rightarrow \quad \frac{1}{6}\,(n+1)(n-1)n(n-3)\,, \tag{50}$$

$$\overset{L}{R}_{\alpha\beta} \quad \rightarrow \quad n^2\,. \tag{51}$$

These pieces have characteristic trace properties

$$e_\alpha \rfloor \mathrm{Weyl}^{\alpha\beta} = e_\alpha \rfloor \overset{P}{R}{}^{\alpha\beta} = 0\,, \quad e_\alpha \rfloor \overset{L}{R}{}^{\alpha\beta} = e_\alpha \rfloor R^{\alpha\beta}\,. \tag{52}$$

By means of those we find $\mathrm{Weyl}^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma} = \overset{P}{R}{}^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma} = 0$. Thus, only the piece $\overset{L}{R}{}^{\alpha\beta}$ contributes to the Einstein 3-form. This is even more apparent by substituting $\vartheta^\gamma\wedge\eta_{\alpha\beta\gamma} = (n-2)\,\eta_{\alpha\beta}$ into eq.(27),

$$G_\alpha \;=\; L^\beta\wedge\eta_{\beta\alpha} = \frac{1}{n-2}\,L^\beta\wedge\vartheta^\gamma\wedge\eta_{\beta\alpha\gamma} = -\frac{1}{n-2}\,\vartheta^{[\beta}\wedge L^{\gamma]}\wedge\eta_{\alpha\beta\gamma}\,,$$

or, by using (46),

$$\boxed{G_\alpha = \frac{1}{2}\overset{L}{R}{}^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma}\,. \qquad (53)}$$

We can use this relation in order to obtain another well motivated representation of the invariant $L^\alpha\wedge G_\alpha$ by rewriting it according to[8]

$$\begin{aligned} I_S \;&:=\; -L^\alpha{}_{[\alpha}L^\beta{}_{\beta]}\,\eta = \frac{1}{2}\,L^\alpha\wedge G_\alpha = \frac{1}{4}\,L^\alpha\wedge\eta_{\alpha\beta\gamma}\wedge\overset{L}{R}{}^{\beta\gamma} \\ &=\; -\frac{n-2}{8(n-3)}\overset{L}{R}{}^{\alpha\beta}\wedge\overset{L}{R}{}^{\gamma\delta}\,\eta_{\alpha\beta\gamma\delta}\,. \end{aligned} \tag{54}$$

In this way, I_S turned out to be one of the basic quadratic invariants[9] (scalar-valued n-forms) of $\overset{L}{R}_{\alpha\beta}$.

[8] We use $(n-3)\,\eta_{\alpha\beta\gamma} = \vartheta^\delta\wedge\eta_{\alpha\beta\gamma\delta}$.

[9] In the case $n=4$ we can define the *Lie-dual* of $\overset{L}{R}_{\alpha\beta}$ by $\overset{L}{R}{}^{*}_{\alpha\beta} := \frac{1}{2}\overset{L}{R}{}^{\gamma\delta}\,\eta_{\alpha\beta\gamma\delta}$. Then I_S reads $I_S = -\frac{1}{2}\overset{L}{R}{}^{\alpha\beta}\wedge\overset{L}{R}{}^{*}_{\alpha\beta}$. Using the irreducible decomposition (46), I_S can also be expressed in terms of the Hodge-dual $I_S \propto R^{\alpha\beta}\wedge{}^\star\left({}^{(4)}R_{\alpha\beta} - {}^{(5)}R_{\alpha\beta} - {}^{(6)}R_{\alpha\beta}\right)$. If $DT^\alpha\equiv 0$, we can use $G_\alpha = {}^\star L_\alpha - L\,\eta_\alpha$ and obtain $I_S = \frac{1}{2}\left[L^\alpha\wedge{}^\star L_\alpha - (L)^2\,\eta\right] = \frac{1}{2}\left[L_\nu{}^\alpha\,L^\nu{}_\alpha - (L)^2\right]\eta$. The positions of the indices differ from those in eq.(40)!

We collect the various representation of the invariant I_S in

$$\boxed{I_S = \frac{1}{2}\, L^\alpha \wedge G_\alpha = -\frac{1}{2}\, L^\alpha \wedge L^\beta \wedge \eta_{\alpha\beta} = -\frac{n-2}{8(n-3)} \overset{L}{R}{}^{\alpha\beta} \wedge \overset{L}{R}{}^{\gamma\delta}\, \eta_{\alpha\beta\gamma\delta}\,. \quad (55)}$$

6 Discussion

In view of our observations, we may put the main result of our investigations as follows. The basic quantity here is the "trace-part" $\overset{L}{R}{}_{\alpha\beta}$ of the curvature with its n^2 independent components. A vector-valued 1-form, a vector-valued $(n-1)$-form, and a 2nd rank tensor valued 0-form, respectively, have the same number of independent components. Thus $\overset{L}{R}{}_{\alpha\beta}$, as displayed in eq.(53) and in eq.(27), can be mapped into a $(n-1)$-form by means of the η-basis, yielding the Einstein 3-form G_α, into a 1-form, yielding the L_α 1-form, and into an $n \times n$ matrix, yielding the Einstein tensor $G_{\alpha\beta}$. Realizing this, makes the algebraic relations between the stated quantities quite clear. These results are represented by Eqs.(27, 53), and (46).

We also would like to mention that eq.(23) which expresses G_α in terms of L_β and eq.(35) which expresses L_α in terms of G_α hold for arbitrary $(n-1)$ forms G_α and 1-forms L_β, respectively. Hence, eq.(23) and eq.(35) establish a *duality relation* between $(n-1)$-forms and 1-forms.

The invariant $L^\alpha{}_{[\alpha}\, L^\beta{}_{\beta]}$, which was derived by Salgado as second principal invariant of $L^\alpha{}_\beta$ (arising in connection with the characteristic polynomial), in our context emerges (i) as one of the the most obvious invariants constructed from L_α and G_α, (ii) as a basic quadratic invariant of L_α, and (iii) as a basic quadratic invariant of the curvature piece $\overset{L}{R}{}_{\alpha\beta}$. These results are displayed in (55) and (29, 39).

We have extensively studied the algebraic properties of L_α. It was quite helpful to check all formulas by means of the Excalc package of the computer algebra system Reduce, see [10].

Also the differential properties of L_α are very interesting. In a Riemannian space we have $D\vartheta^\alpha = 0$. By using (47), we can represent the second Bianchi Identity as follows

$$D\mathrm{Weyl}_{\alpha\beta} = -\frac{2}{n-2}\, \vartheta_{[\alpha} \wedge C_{\beta]}\,, \tag{56}$$

where we defined the Cotton 2-form by

$$C_\alpha := DL_\alpha\,. \tag{57}$$

In this way, L_α appears as *potential* of the Cotton 2-form. Since the conformal (Weyl-) curvature is tracefree, the (twice) contracted 2nd Bianchi identity reads

$$0 = e_\alpha \rfloor C^\alpha = e_\alpha \rfloor DL^\alpha\,. \tag{58}$$

The Cotton 2-form, especially its relation to the conformal properties of spacetime, is subject of a current project [4]. The definition (57) of the Cotton 2-form can be transferred to a Riemann-Cartan space.

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7 Appendix

7.1 Some relations for the exterior and interior products

In order to avoid dimension-dependent signs, it is of special importance to take care of the order of the forms in the exterior products. We would like to remind the reader of the following relations which hold for a p-form ϕ and a q-form ψ:

$$\phi \wedge \psi = (-1)^{pq}\, \psi \wedge \phi\,, \tag{59}$$

$$e_\mu \rfloor (\phi \wedge \psi) = (e_\mu \rfloor \phi) \wedge \psi + (-1)^p\, \phi \wedge (e_\mu \rfloor \psi)\,. \tag{60}$$

7.2 The variational derivative with respect to p-forms

The variation of a function F which depends on a p-form ψ is defined to be

$$\delta F := F(\psi + \delta\psi) - F(\psi)\,, \tag{61}$$

where the p-form $\delta\psi$ is supposed to be an arbitrary "small" deviation. With given F, we can elementary evaluate the right-hand side of eq.(61). We then neglect all terms of quadratic and higher order in $\delta\psi$ and bring the result into the form

$$\delta F = \delta\psi \wedge (\,\dots)\,. \tag{62}$$

The expression in the parentheses is defined to be the partial derivative with respect to ψ. This prescription especially fixes the sign. The generalization to an arbitrary number of forms or tensor-valued forms is straightforward.

Due to the definition, the variation obeys a Leibniz-rule

$$\begin{aligned} \delta(\phi \wedge \psi) &\overset{(61)}{=} ((\phi + \delta\phi) \wedge (\psi\ + \delta\psi)) - \phi \wedge \psi \\ &= \phi \wedge \delta\psi + \delta\phi \wedge \psi + \underbrace{\delta\phi \wedge \delta\psi}_{\hookrightarrow 0} \\ &= \phi \wedge \delta\psi + \delta\phi \wedge \psi\,. \end{aligned} \tag{63}$$

The variational derivative can be introduced in the usual way. However, in this context we just note that in the case in which F does not depend on the derivatives $d\psi$, the partial and the variational derivative coincide.

7.3 Some relations for the Hodge-star

We frequently made use of the following relations for the Hodge-star. ψ an ϕ are two p-forms of the same degree, a, $b \in \mathbf{R}$ are numbers.

$$^\star(a\psi + b\phi) = a^\star\psi + b^\star\phi\,, \qquad \text{H1}\,. \tag{64}$$

$$^{\star\star}\psi = (-1)^{p(n-p)+\text{ind}}\,\psi\,, \qquad \text{H2}\,, \tag{65}$$

where ind denotes the number of negative Eigenvalues of the metric, 3 in the case of a $(3+1)$-dimensional spacetime.

$$^\star(e_\alpha \rfloor \psi) = (-1)^{(p-1)}\,\vartheta_a \wedge {}^\star\psi\,, \qquad \text{H3}\,. \tag{66}$$

$$e_\alpha \rfloor {}^\star\psi = {}^\star(\psi \wedge \vartheta_\alpha)\,, \qquad \text{H4}\,. \tag{67}$$

$$^\star\psi \wedge \phi = {}^\star\phi \wedge \psi\,, \qquad \text{H5}\,. \tag{68}$$

7.4 Variation of L_α

We substitute eq.(35) in eq.(36):

$$\delta L^\alpha = L^\alpha(\delta G_\beta) = (-1)^{\text{ind}}\,{}^\star\left[e^\beta \rfloor (\delta G_\beta \wedge \vartheta^\alpha) - \frac{1}{n-1}\,e^\alpha \rfloor (\delta G_\beta \wedge \vartheta^\beta)\right]\,. \tag{69}$$

The expression in the square-brackets is a $(n-1)$-form. By H5 we then have

$$\begin{aligned} \delta L^\alpha \wedge G_\alpha &= (-1)^{\text{ind}}\,{}^\star\left[e^\beta \rfloor (\delta G_\beta \wedge \vartheta^\alpha) - \frac{1}{n-1}\,e^\alpha \rfloor (\delta G_\beta \wedge \vartheta^\beta)\right] \wedge G_\alpha \\ &= (-1)^{\text{ind}}\,\left\{{}^\star G_\alpha \wedge \left[e^\beta \rfloor (\delta G_\beta \wedge \vartheta^\alpha)\right] - \frac{1}{n-1}\,{}^\star G_\alpha \wedge \left[e^\alpha \rfloor (\delta G_\beta \wedge \vartheta^\beta)\right]\right\}\,. \end{aligned} \tag{70}$$

$^\star G_\alpha$ is a 1-form and $\delta G_\beta \wedge \vartheta^\beta$, $\delta G_\beta \wedge \vartheta^\beta$ are n-forms. Thus

$$\begin{aligned} 0 &= e^\beta \rfloor \left[{}^\star G_\alpha \wedge (\delta G_\beta \wedge \vartheta^\alpha)\right] = (e^\beta \rfloor {}^\star G_\alpha)\,\delta G_\beta \wedge \vartheta^\alpha - {}^\star G_\alpha \wedge \left[e^\beta \rfloor (\delta G_\beta \wedge \vartheta^\alpha)\right]\,, \\ 0 &= e^\alpha \rfloor \left[{}^\star G_\alpha \wedge (\delta G_\beta \wedge \vartheta^\beta)\right] = (e^\alpha \rfloor {}^\star G_\alpha)\,\delta G_\beta \wedge \vartheta^\beta - {}^\star G_\alpha \wedge \left[e^\alpha \rfloor (\delta G_\beta \wedge \vartheta^\beta)\right]\,. \end{aligned}$$

Substituting this into eq.(70) yields

$$\begin{aligned} \delta L^\alpha \wedge G_\alpha &= (-1)^{\text{ind}}\,(-1)^{(n-1)}\left[(e^\beta \rfloor {}^\star G_\alpha)\,\vartheta^\alpha - \frac{1}{n-1}\,(e^\alpha \rfloor {}^\star G_\alpha)\,\vartheta^\beta\right] \wedge \delta G_\beta \\ &= (-1)^{\text{ind}}\,(-1)^{(n-1)}\left[{}^\star(G_\alpha \wedge \vartheta^\beta)\,\vartheta^\alpha - \frac{1}{n-1}\,{}^\star(G_\alpha \wedge \vartheta^\alpha)\,\vartheta^\beta\right] \wedge \delta G_\beta \\ &= (-1)^{\text{ind}}\,{}^\star\left[e^\alpha \rfloor (G_\alpha \wedge \vartheta^\beta) - \frac{1}{n-1}\,e^\beta \rfloor (G_\alpha \wedge \vartheta^\alpha)\right] \wedge \delta G_\beta \\ &= L^\beta \wedge \delta G_\beta \\ &= (-1)^{(n-1)}\,\delta G_\beta \wedge L^\beta\,. \end{aligned} \tag{71}$$

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